OSCILLATIONS IN THE WALRASIAN GENERAL EQUILIBRIUM THEORY WITH ENDOGENOUS WEALTH AND HUMAN CAPITAL ACCUMULATION

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Abstract
This paper generalizes the dynamic growth model with wealth accumulation and human capital accumulation proposed by Zhang (2013) by making all the parameters as time-dependent parameters. The original model is an extension of the Uzawa-Lucas model to a heterogeneous household economy with multiple ways of human capital accumulation. It synthesizes the basic ideas of the Walrasian general equilibrium theory, Arrow's learning by producing, Zhang's learning by consuming (creative learning), the neoclassical growth theory, and the Uzawa-Lucas two-sector model. The behavior of the household is described with an alternative approach to household behavior. The economic system consists of one production sector and one education sector. Households are different in propensities to save, to obtain education and to consume, and in learning abilities. We simulate the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks.

Keywords: wealth; human capital; heterogeneous households; exogenous perturbation; and business oscillations.

JEL. Classification: *O41*

I. INTRODUCTION

Economic fluctuations or business cycles are commonly observed in empirical studies. Some researches consider economic oscillations as exogenous. A typical example is agricultural production which is influenced by seasonal changes as well as long-term global climates. As modern nonlinear dynamic economic theory shows that business cycles may also occur in a self-organized economic system without any exogenous influences. There are a lot of theoretical and empirical research about mechanisms and phenomena of economic fluctuations (e.g., Zhang, 1991, 2005, 2006; Lorenz, 1993; Flaschel *et al* 1997; Chiarella and Flaschel, 2000; Shone, 2002; Gandolfo, 2005; Puu, 2011; Nolte, 2015). These studies show that business cycles, regular as well as irregular oscillations, and chaos exist in different economic models. Different studies explain economic business cycles from different perspectives. Lucas (1977) demonstrates how some shocks affect all sectors in an economy. Chatterjee and Ravikumar (1992) build a neoclassical growth model with seasonal perturbations to taste and technology. The economic system reacts to seasonal demand and supply perturbations with fluctuations. Gabaix (2011) tries to show that uncorrelated sectoral shocks are determinants of aggregate fluctuations (see also, Giovanni, et al. 2014; Stella, 2015). This study attempts to identify economic fluctuations due to exogenous shocks.

This paper is to introduce exogenous shocks to a growth model proposed by Zhang (2013). Zhang's model deals with dynamic interdependence between economic growth and human capital with education. It has become evident that the neoclassical growth theory needs to be extended in order to explain why countries grow differently. Education has been identified as one of important determinants of economic growth (Easterlin, 1981; Hanushek and Kimko, 2000; Barro, 2001; Krueger and Lindahl, 2001; Castelló-Climent and Hidalgo-Cabrillana, 2012; and Barro and Lee, 2013). There are many studies about dynamic interdependence between education and economic growth. Mincer (1974) published the seminal work in 1974 on the impact of education on earnings. Tilak (1989) argues that spread education can substantially reduce inequality within countries. Could *et al*. (2001) examine the evolution of wage inequality within and between industries and education groups in the past few decades. The study concludes that increasing randomness is the primary source of inequality growth within uneducated workers, but inequality growth within educated workers is determined more by changes in the composition and return to ability. Tselios (2008) studies the relationship between income and educational inequalities in the regions of the European Union, finding a positive relationship between income and educational inequalities. Fleisher *et al.* (2011) examine the role of education on worker productivity and firms' total factor productivity on the basis of firm-level data from China, concluding that an additional year of schooling raises marginal product by 30.1 percent. Zhu (2011) studies the individual heterogeneity in returns to education in China from 1995-2002. The study provides heterogeneous effects both within and between gender groups. Zhu finds that the heterogeneity in schooling returns falls from 1995 to 2002 for both genders in urban China, although their rates of education return have increased substantially. There is also a large number of the theoretical literature on human capital, knowledge and economic growth. The literature has increasingly expanded since Romer (1986) re-examined issues of endogenous technological change and economic growth in his 1986's paper (see also, Lucas, 1988; Grossman and Helpman, 1991; Parente and Prescott, 1994;

Aghion and Howitt, 1998; Thompson, 2001; Luttmer, 2011; Bowlus and Robinson, 2012; Benhabib, 2014; Lucas and Moll, 2014; Stokey, 2015). The first formal dynamic growth model with education was proposed by Uzawa (1965) and further developed by Lucas (1988). Nevertheless, a main problem in the Uzawa-Lucas model and many of their extensions and generalizations is that all skills and human capital are formed due to formal schooling. Nevertheless, much of human capital may be accumulated in family and many other social and economic activities. In addition to formal schooling, this study takes account of Arrow's learning by doing (Arrow, 1962) and Zhang's creative leisure (Zhang, 2007) in modeling human capital accumulation.

In almost all formal models of economic growth and education, the population is assumed homogenous. Nevertheless, different households have different propensities to save and to receive education, and have different abilities in absorbing knowledge and increasing human capital through education, learning by doing and learning by consuming. There are a few approaches with endogenous human capital and heterogeneous households (e.g., Galor and Zeira, 1993; Maoz and Moav, 1999; Fender and Wang, 2003; Cardak, 2004; Erosa *et al*. 2010). A main deviation of our approach from the previous models is that we derive demand of education in an alternative approach to the typical Ramsey approach. We also introduce endogenous wealth accumulation. This allows us to explicitly derive the differential equations of the economic system and simulate transition processes. Moreover, we also include endogenous wealth accumulation in our model. This paper is built on Zhang's model (Zhang, 2013). The model is built upon the Walrasian general equilibrium theory and the three main growth models – the neoclassical two-sector growth model, Arrow's learning by doing model, and the Uzawa-Lucas's growth model with education - in the growth literature. The main mechanisms of economic structure and growth in these theories are integrated into a single framework. The main generalization of this study is to treat all the time-independent parameters in Zhang's model as time-dependent. This will make Zhang's original model far more robust as there are many factors, such as technological change, institutional shifts, fashions, seasonal factors, are time-dependent and are considered exogenous. This paper is concerned with identifying economic fluctuation in the model. The rest of the paper is organized as follows. Section 2 defines the basic model. Section 3 shows how we solve the dynamics and simulates the motion of the global economy. Section 4 carries out comparative dynamic analysis to examine the impact of changes in some parameters on the motion of the global economy. Section 5 concludes the study. The appendix proves the main results in Section 3.

II. THE BASIC MODEL

The model is based on Zhang (2013). We generalize Zhang's model mainly by consider all the time-independent parameters as being time-dependent. The economy a combination of Uzawa's two sector model and Uzawa-Lucas' two sectors model. It has one education, one capital good, and one consumer goods sectors. Most aspects of the production sectors are similar to Uzawa's two-sector growth model (Uzawa, 1961; Burmeister and Dobell, 1970; Azariadis, 1993; and Barro and Sala-i-Martin, 1995). The education sector is based on the Uzawa-Lucas two-sector model. Households own assets of the economy and distribute their incomes to receive education, to consume and to save. Firms use labor and physical capital inputs to supply goods and services. Exchanges take place in perfectly competitive markets. Factor markets work well and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production, labor, managerial skill and capital ownership. We classify the population into J groups, basing on their preference and human capital. Group j 's population is denoted by $\bar{N}_j(t)$, ($j = 1, \ldots, J$). We measure prices in terms of the commodity and the price of the commodity be unit. We use $p_s(t)$ to stand for the price of consumer good at time t. We denote wage and interest rates by $w_j(t)$ and $r(t)$, respectively. We use $H_j(t)$ to stand for group j 's level of human capital. Subscript index, i, s, and e, to stand for capital goods, consumer good, and education sectors, respectively. We use $N_m(t)$ and $K_m(t)$ to represent the labor force and capital stocks employed by sector m. Let $T_j(t)$ and $T_{je}(t)$ stand for, respectively, the work time and study time of a typical worker in group j. The variable $N(t)$ represents the total qualified labor force. A worker's labor force is $T_j(t)H_j^{m_j(t)}(t)$, where $m_j(t)$ is a parameter measuring utilization efficiency of human capital by group j. The labor input is the work time by the effective human capital. A group's labor input is the group's population by each member the labor force, that is, $T_j(t)H_j^{m_j(t)}(t)\bar{N}_j(t)$. The total qualified labor force is the sum of all the groups' labor forces

$$N(t) = \sum_{j=1}^{J} T_j(t)H_j^{m_j(t)}(t)\bar{N}_j(t), \quad j = 1, \ldots, J. \tag{1}$$

Full employment of labor and capital

The total labor force is employed by the three sectors. The labor is fully employed

$$N_i(t)+N_s(t)+N_e(t)=N(t). \qquad (2)$$

The total capital stock $K(t)$ is allocated among the three sectors

$$K_i(t)+K_s(t)+K_e(t)=K(t). \qquad (3)$$

We use $\bar{k}_j(t)$ to stand for per capita wealth of group j at t. Group j 's wealth is $\bar{k}_j(t)\bar{N}_j(t)$. As wealth is help by the households, we have

$$K(t)=\sum_{j=1}^{J}\bar{k}_j(t)\bar{N}_j(t). \qquad (4)$$

The capital goods sector

We use $F_m(t)$ to represent the production function of sector m, $m=i, s$. The production function of the capital goods sector is specified as follows

$$F_i(t)=A_i(t)K_i^{\alpha_i(t)}(t)N_i^{\beta_i(t)}(t),\ \ \alpha_i(t),\ \beta_i(t)>0,\ \ \alpha_i(t)+\beta_i(t)=1, \qquad (5)$$

where $A_i(t)$, $\alpha_i(t)$, and $\beta_i(t)$ are positive parameters. The marginal conditions are

$$r(t)+\delta_k(t)=\frac{\alpha_i(t)F_i(t)}{K_i(t)},\ \ w(t)=\frac{\beta_i(t)F_i(t)}{N_i(t)}. \qquad (6)$$

The consumer goods sector

The production function of the consumer goods sector is specified as follows

$$F_s(t)=A_s(t)K_s^{\alpha_s(t)}(t)N_s^{\beta_s(t)}(t),\ \ \alpha_s(t)+\beta_s(t)=1,\ \ \alpha_s(t),\ \beta_s(t)>0, \qquad (7)$$

where $A_s(t)$, $\alpha_s(t)$, and $\beta_s(t)$ are the parameters. The marginal conditions are given by

$$r(t)+\delta_k(t)=\frac{\alpha_s(t)p_s(t)F_s(t)}{K_s(t)},\ \ w(t)=\frac{\beta_s(t)p_s(t)F_s(t)}{N_s(t)}. \qquad (8)$$

Education sector

Education is perfectly competitive. Students pay the education fee $p_e(t)$ per unit of time. The education sector pays teachers and capital with the market rates. We specify the production function of the education sector as follows

$$F_e(t)=A_e K_e^{\alpha_e(t)}(t)N_e^{\beta_e(t)}(t),\ \ \alpha_e(t),\ \beta_e(t)>0,\ \ \alpha_e(t)+\beta_e(t)=1, \qquad (9)$$

where $A_e(t)$, $\alpha_e(t)$ and $\beta_e(t)$ are positive parameters. The marginal conditions are

$$r(t)+\delta_k(t)=\frac{\alpha_e(t)p_e(t)F_e(t)}{K_e(t)},\ \ w(t)=\frac{\beta_e(t)p_e(t)F_e(t)}{N_e(t)}. \qquad (10)$$

Consumer behaviors and wealth dynamics

Consumers make decisions on choice of consumption levels of education, services and commodities as well as on how much to save. There are different models about decisions on education (Becker, 1981; Cox, 1987; Behrman *et al.* 1982; Fernandez and Rogerson, 1998; Banerjee, 2004; Florida, et al. 2008; Galindev, 2011). According to Chen and Chevalier (2008), "Making and exploiting an investment in human capital requires individuals to sacrifice not only consumption, but also leisure. When estimating the returns to education, existing studies typically weigh the monetary costs of schooling (tuition and forgone wages) against increased wages, neglecting the associated labor/leisure tradeoff." We will overcome this shortcoming by introducing endogenous time distribution. This study applies the approach by Zhang (1993) to model behavior of households. Let $\bar{k}_j(t)$ stand for the per capita wealth of group j. Introduce $\bar{K}_j(t) = \bar{k}_j(t)\bar{N}_j(t)$. The current income is the sum of the interest payment $r(t)\bar{k}_j(t)$ and the wage payment $T_j(t)w_j(t)$

$$y_j(t) = r(t)\bar{k}_j(t) + T_j(t)w_j(t).$$

The per capita disposable income is

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = (1 + r(t))\bar{k}_j(t) + T_j(t)w_j(t). \tag{11}$$

The disposable income is used for saving, consumption, and education. The typical consumer distributes the total available budget among saving $s_j(t)$, consumption of consumer goods $c_j(t)$, and education $p_e(t)T_{je}(t)$. The budget constraint is

$$p_s(t)c_j(t) + s_j(t) + p_e(t)T_{je}(t) = \hat{y}_j(t) = (1 + r(t))\bar{k}_j(t) + w_j(t)T_j(t), \tag{12}$$

The available time is distributed between work, leisure and education

$$T_j(t) + T_{je}(t) + \tilde{T}_j(t) = T_0, \tag{13}$$

where $\tilde{T}_j(t)$ is the leisure time of the representative household and T_0 is the total available time. Substitute (13) into (12)

$$w_j(t)\tilde{T}_j(t) + p_s(t)c_j(t) + s_j(t) + p_j(t)T_{je}(t) = \bar{y}_j(t) \equiv (1 + r(t))\bar{k}_j(t) + T_0 w_j(t), \tag{14}$$

where

$$p_j(t) \equiv p_e(t) + w_j(t).$$

As education increases human capital, a rise in education tends to result in higher wages (e.g., Heckman, 1976; Lazear, 1977; Malchow-Møller, *et al.* 2011). As Lazear (1977: 570) points out: "education is simply a normal consumption good and that, like all other normal goods, an increase in wealth will produce an increase in the amount of schooling purchased. Increased incomes are associated with higher schooling attainment as the simple result of an income effect." Education also brings about direct pleasure, more knowledgeable, higher social status and so on. We assume that the consumer's utility function is dependent on $\bar{T}_j(t)$, $T_{je}(t)$, $c_j(t)$, and $s_j(t)$ as follows

$$U(t) = \bar{T}_j^{\sigma_{j0}(t)}(t)T_{je}^{\eta_{j0}(t)}(t)c^{\xi_{j0}(t)}(t)s^{\lambda_{j0}(t)}(t), \ \ \sigma_{j0}(t), \xi_{j0}(t), \lambda_{j0}(t), \eta_{j0}(t) > 0, \tag{15}$$

where $\sigma_{j0}(t)$ is the propensity to use leisure time, $\eta_{j0}(t)$ the propensity to obtain education, $\xi_{j0}(t)$ is the propensity to consume, and $\lambda_{j0}(t)$ the propensity to own wealth. Maximizing $U_j(t)$ subject to (9) yields

$$\bar{T}_j(t) = \frac{\sigma_j(t)\bar{y}_j(t)}{w_j(t)}, \ \ T_{je}(t) = \frac{\eta_j(t)\bar{y}_j(t)}{p_j(t)}, \ \ c_j(t) = \frac{\xi_j(t)\bar{y}_j(t)}{p_s(t)}, \ \ s_j(t) = \lambda_j(t)\bar{y}_j(t), \tag{16}$$

where

$$\sigma_{j0}(t) \equiv \rho_j(t)\sigma_{j0}(t), \ \ \eta_{j0}(t) \equiv \rho_j(t)\eta_{j0}(t), \ \ \xi_{j0}(t) \equiv \rho_j(t)\xi_{j0}(t), \ \ \lambda_{j0}(t) \equiv \rho_j(t)\lambda_{j0}(t),$$
$$\rho_j(t) \equiv \frac{1}{\sigma_{j0}(t) + \xi_{j0}(t) + \lambda_{j0}(t) + \eta_{j0}(t)}.$$

According to the definitions of $s_j(t)$, the wealth accumulation of the representative household in group j is given by

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t) - \frac{\dot{\bar{N}}_j(t)\bar{k}_j(t)}{\bar{N}_j(t)}. \qquad (17)$$

This equation simply states that the change in wealth is equal to savings minus dissaving.

Dynamics of human capital

We take account of three sources of improving human capital, through education, "learning by producing", and "learning by leisure". Arrow (1962) first introduced learning by doing into growth theory; Uzawa (1965) took account of trade-offs between investment in education and capital accumulation, and Zhang (2007) introduced impact of consumption on human capital accumulation (via the so-called creative leisure) into growth theory. Basing on Zhang (2013), we have

$$\dot{H}_j(t) = \frac{\upsilon_{je}(t)F_e^{a_{je}(t)}(t)\left(H_j^{m_j(t)}(t)T_{je}(t)\bar{N}_j\right)^{b_{je}(t)}}{H_j^{\pi_{je}(t)}(t)\bar{N}_j(t)} + \frac{\upsilon_{ji}(t)F_i^{a_{ji}}(t)}{H_j^{\pi_{ji}(t)}(t)\bar{N}_j(t)} + \frac{\upsilon_{jh}(t)C_j^{a_{jh}(t)}(t)}{H_j^{\pi_h(t)}(t)\bar{N}_j(t)} - \delta_{jh}(t)H_j(t), \qquad (18)$$

where $\delta_{jh}(t)\,(>0)$ is the depreciation rate of human capital, $\upsilon_{je}(t)$, $\upsilon_{ji}(t)$, $\upsilon_{jh}(t)$, $a_{je}(t)$, $b_{je}(t)$, $a_{ji}(t)$, and $a_{jh}(t)$ are non-negative parameters. The signs of the parameters $\pi_{je}(t)$, $\pi_{ji}(t)$, and $\pi_{jh}(t)$ are not specified as they may be either negative or positive. The above equation is a synthesis and generalization of Arrow's, Uzawa's, and Zhang's ideas about human capital accumulation. The first term in the right-hand of (18) describes the contribution to human capital improvement through education. We take account of learning by producing effects in human capital accumulation by the term $\upsilon_{ji} F_i^{a_{ji}} / H_j^{\pi_{ji}}$. We take account of learning by consuming by the term $\upsilon_{jh} C_j^{a_{jh}} / H_j^{\pi_{jh}} \bar{N}_j$.

Demand of and supply for education

The condition that the demand for and supply of education balances at any point of time implies

$$\sum_{j=1}^{J} T_{je}(t)\bar{N}_j = F_e(t). \qquad (19)$$

Demand of and supply for consumer good

The condition that the total demand is equal to the total supply implies

$$\sum_{j=1}^{J} c_j(t)\bar{N}_j(t) = F_s(t). \qquad (20)$$

Demand of and supply for capital goods

The output should equal the depreciation of capital stock and the net savings. We have

$$\sum_{j=1}^{J} s_j(t)\bar{N}_j(t) - K(t) + \delta_k K(t) = F_i(t). \qquad (21)$$

We completed the model. The model is structurally general in the sense that some well-known models in economics can be considered as its special cases. For instance, if we fix wealth and human capital and allow the number of types of households equal the population, then the model is a Walrasian general equilibrium model. If the population is homogeneous, our model is structurally similar to the neoclassical growth model by Solow (1956) and Uzawa (1961). It is structurally similar to the multi-class models by Pasinetti and Samuelson (e.g., Samuelson, 1959; Pasinetti, 1960, 1974). Obviously, if both human capital and physical capital are constant, the model is a Walrasian general equilibrium model. We now examine dynamics of the model.

III. THE DYNAMICS AND ITS PROPERTIES

As the system consists of any number of types of households, its dynamics may be highly dimensional. The following lemma shows that the economic dynamics is represented by $2J$ dimensional differential equations.

Lemma

The dynamics of the economy is governed by the following $2J$ dimensional differential equations system with $z(t), \ \{\bar{k}_j(t)\}, \ (H_j(t))$, where $\{\bar{k}_j(t)\} \equiv (\bar{k}_2(t), \cdots, \bar{k}_J(t))$ and $(H_j(t)) \equiv (H_1(t), \cdots, H_J(t))$, as the variables

$$\dot{z}(t) = \Lambda_1\left(z(t), \left(H_j(t)\right), \{\bar{k}_j(t)\}, t\right),$$
$$\dot{\bar{k}}_j(t) = \Lambda_j\left(z(t), \left(H_j(t)\right), \{\bar{k}_j(t)\}, t\right), \quad j = 2, ..., J,$$
$$\dot{H}_j(t) = \Omega_j\left(z(t), \left(H_j(t)\right), \{\bar{k}_j(t)\}, t\right), \quad j = 1, ..., J,$$

in which Λ_j and Ω_j are unique functions of $z(t), \ \{\bar{k}_j(t)\}, \ (H_j(t))$, and t at any point in time, defined in Appendix. For any given positive values of $z(t), \ \{\bar{k}_j(t)\}$, and $(H_j(t))$ at any point in time, the other variables are uniquely determined by the following procedure: $r(t)$ and $w(t)$ by (A3) → $w_j(t)$ by (A4) → $p_e(t)$ and $p_s(t)$ by (A5) → $\bar{k}_1(t)$ by (A18) → $N_i(t)$ and $N_e(t)$ by (A14) → $N(t)$ by (A11) → $N_s(t)$ by (A8) → $\bar{y}_j(t)$ by (A6) → $K_m(t)$ (A1) → $F_i(t), \ F_s(t)$, and $F_e(t)$ by the definitions → $\bar{T}_j(t), \ c_j(t), \ T_{je}(t)$, and $s_j(t)$ by (16) → $K(t)$ by (3).

We have the dynamic equations for the economy with any number of types of households. The system is nonlinear and is of high dimension. We simulate the model. To illustrate motion of the system, following Zhang (2013) we first assume the parameters to be time-independent and specify the parameters as follows

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \\ 60 \end{pmatrix}, \ \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.4 \\ 0.3 \end{pmatrix}, \ \begin{pmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.18 \\ 0.2 \end{pmatrix}, \ \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.75 \\ 0.7 \end{pmatrix}, \ \begin{pmatrix} \eta_{10} \\ \eta_{20} \\ \eta_{30} \end{pmatrix} = \begin{pmatrix} 0.015 \\ 0.010 \\ 0.008 \end{pmatrix},$$
$$\begin{pmatrix} \sigma_{10} \\ \sigma_{20} \\ \sigma_{30} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.23 \\ 0.2 \end{pmatrix}, \ \begin{pmatrix} v_{1e} \\ v_{2e} \\ v_{3e} \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.5 \end{pmatrix}, \ \begin{pmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2 \\ 1.7 \end{pmatrix}, \ \begin{pmatrix} v_{1h} \\ v_{2h} \\ v_{3h} \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.5 \end{pmatrix}, \ \begin{pmatrix} a_{1e} \\ a_{2e} \\ a_{3e} \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.45 \end{pmatrix},$$
$$\begin{pmatrix} b_{1e} \\ b_{2e} \\ b_{3e} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.55 \\ 0.6 \end{pmatrix}, \ \begin{pmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.45 \\ 0.5 \end{pmatrix}, \ \begin{pmatrix} a_{1h} \\ a_{2h} \\ a_{3h} \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.15 \\ 0.2 \end{pmatrix}, \ \begin{pmatrix} b_{1h} \\ b_{2h} \\ b_{3h} \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.35 \\ 0.4 \end{pmatrix}, \ \begin{pmatrix} \pi_{1e} \\ \pi_{2e} \\ \pi_{3e} \end{pmatrix} = \begin{pmatrix} -0.2 \\ -0.15 \\ -0.1 \end{pmatrix},$$
$$\begin{pmatrix} \pi_{1i} \\ \pi_{2i} \\ \pi_{3i} \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.75 \\ 0.8 \end{pmatrix}, \ \begin{pmatrix} \pi_{1h} \\ \pi_{2h} \\ \pi_{3h} \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.15 \\ 0.2 \end{pmatrix}, \quad \delta_{1h} = 0.04, \ \delta_{2h} = 0.05, \ \delta_{3h} = 0.06,$$

$$A_i = 0.9,\ A_s = A_e = 0.8,\ \alpha_i = 0.32,\ \alpha_s = 0.34,\ \alpha_e = 0.37,\ T_0 = 1,\ \delta_k = 0.05. \quad (22)$$

We call the three groups respectively as rich, middle, and poor class (RC, MC, PC). We specify the values of the parameters, α_j, in the Cobb-Douglas productions approximately equal to 0.3. The depreciation rates of physical capital and knowledge are specified about 0.05. The RC's propensity to save is 0.8 and the PC's propensity to save is 0.7. The value of the MC's propensity is between the other groups. The RC's propensity to obtain education is highest among the three classes; the PC has the lowest propensity to obtain education. In Figure 1, we plot the motion of the system with the following initial conditions

$$z(0) = 0.09,\ \bar{k}_2(0) = 3,\ \bar{k}_3(0) = 2,\ H_1(0) = 9.3,\ H_2(0) = 3.2,\ H_3(0) = 1.4. \quad (23)$$

In Figure 1, the national output is defined as

$$Y(t) = F_i(t) + p_s(t)F_s(t) + p_e(t)F_e(t).$$

The three groups all increase education time over time. The RC's and PC's levels of human capital are reduced, while the PC's level of human capital is increased.

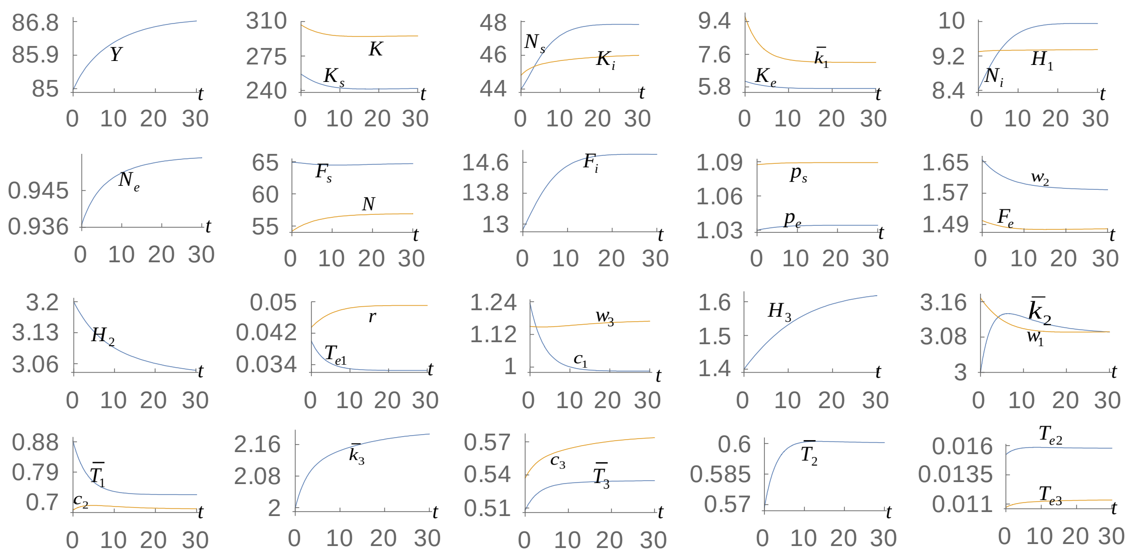

Figure 1. The Motion of Some Variables

The equilibrium values are listed in (24).

$$\begin{gathered} r = 0.05,\ p_s = 1.09,\ p_e = 1.03,\ N = 56.95,\ K = 295.8,\ F_i = 14.8,\ F_s = 64.8, \\ F_e = 1.48,\ N_i = 9.94,\ N_s = 46.1,\ N_e = 0.95,\ K_i = 47.8,\ K_s = 242.3,\ K_e = 5.72, \\ w_1 = 3.1,\ w_2 = 1.6,\ w_3 = 1.2,\ H_1 = 9.35,\ H_2 = 3,\ H_3 = 1.6,\ \bar{k}_1 = 7.2, \\ \bar{k}_2 = 3.1,\ \bar{k}_3 = 2.2,\ T_{1e} = 0.03,\ T_{2e} = 0.02,\ T_{3e} = 0.01,\ \bar{T}_1 = 0.7,\ \bar{T}_2 = 0.6,\ \bar{T}_3 = 0.5, \\ T_1 = 0.25,\ T_2 = 0.38,\ T_3 = 0.45,\ c_1 = 0.99,\ c_2 = 0.68,\ c_3 = 0.58. \end{gathered} \quad (24)$$

It is straightforward to calculate the six eigenvalues as follows

$$-0.33,\ -0.30,\ -0.21,\ -0.1,\ -0.07,\ -0.04.$$

As all the eigenvalues are negative, we see that the equilibrium point is locally stable.

IV. COMPARATIVE DYNAMIC ANALYSIS

We now study effects of changes in some parameters on the motion of the economic system. Zhang (2013) shows how the system reacts to a once-for-all change in parameters. This section shows how the system reacts to time-dependent changes in parameters. For convenience we consider the parameters in (22) as the long-term average values. We make small perturbations around these long-term values. In this study we use $\overline{\Delta} x_j(t)$ to stand for the change rate of the variable $x_j(t)$ due to changes in a parameter value.

Perturbations in the RC' propensity to receive education

First, we examine effects of the following exogenous fluctuations in the RC's propensity to obtain education η_{10} on the economic system

$$\eta_{10}(t) = 0.015 + 0.005\sin(t).$$

The simulation results are plotted in Figure 2. From Figure 2 we see that as the RC changes the propensity to obtain education, the RC's level of human capital fluctuates. The TC's time distribution fluctuates and the other two groups' are slightly affected. The national output is strongly oscillatory. The total labor fluctuates slightly but the labor distribution fluctuates strongly. The output levels of the capital sector and the education are strongly affected and the output of the consumer goods sector is weakly affected.

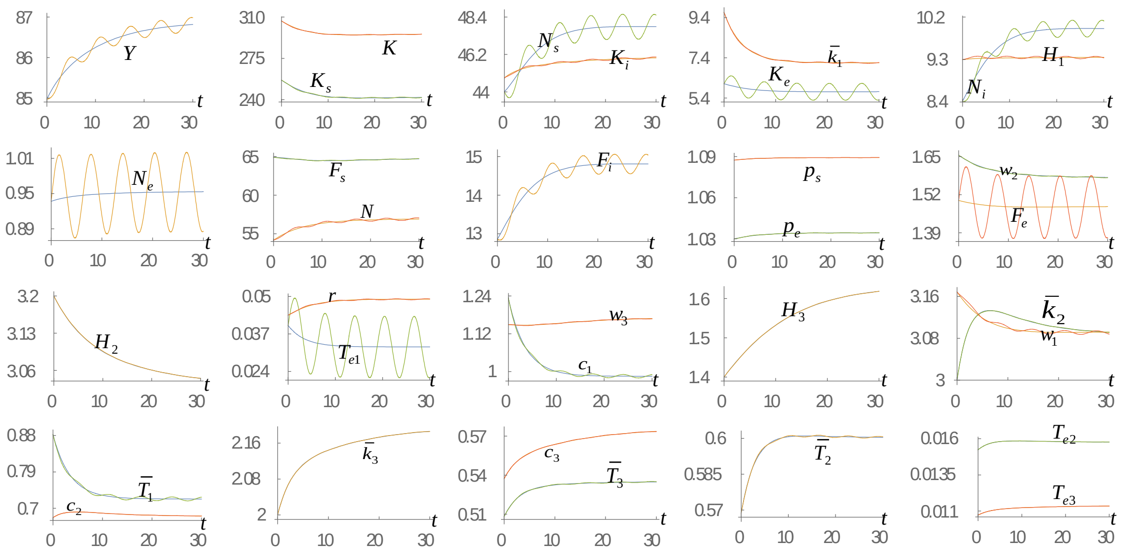

Figure 2. Perturbations in the RC's Propensity to Receive Education

Fluctuations in the RC's propensity to save

We now increase the RC's propensity to save in the following way:

$$\lambda_{10}(t) = 0.8 + 0.03\sin(t).$$

The simulation results are plotted in Figure 3. As the RC's propensity to save is oscillated, the wealth per capita of the class is perturbed but only weakly. The parameter changes have strong effects on the level of the capital sector. The national output, total labor supply and labor distribution fluctuate strongly. The wage rates fluctuate and the prices change slightly.

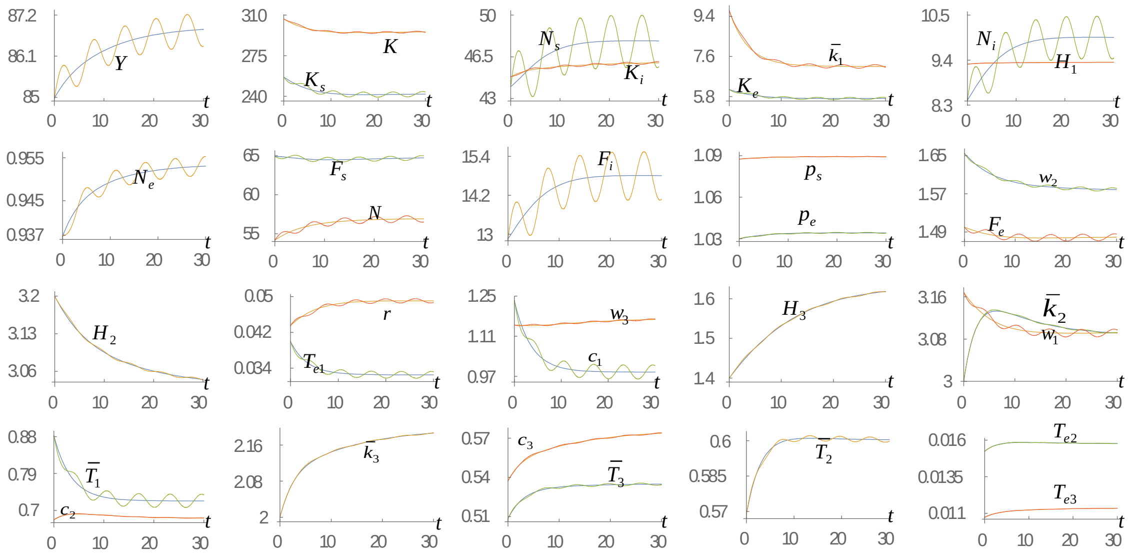

Figure 3. Fluctuations in the RC's Propensity to Save

Fluctuations in the RC's population

We now fluctuate the RC's population in the following way

$$\bar{N}_1(t) = 10 + 0.5\sin(t).$$

The simulation results are plotted in Figure 4. The fluctuations in the RC's population have slight impact on the human capital levels. The RC's wealth per capita fluctuates and the other two classes' wealth levels are slightly affected. The total output, total labor supply and labor distribution fluctuate strongly. The RC's wage rate is perturbed strongly and the other two classes' wage rates fluctuate slightly. The total wealth and capital distribution are weakly affected.

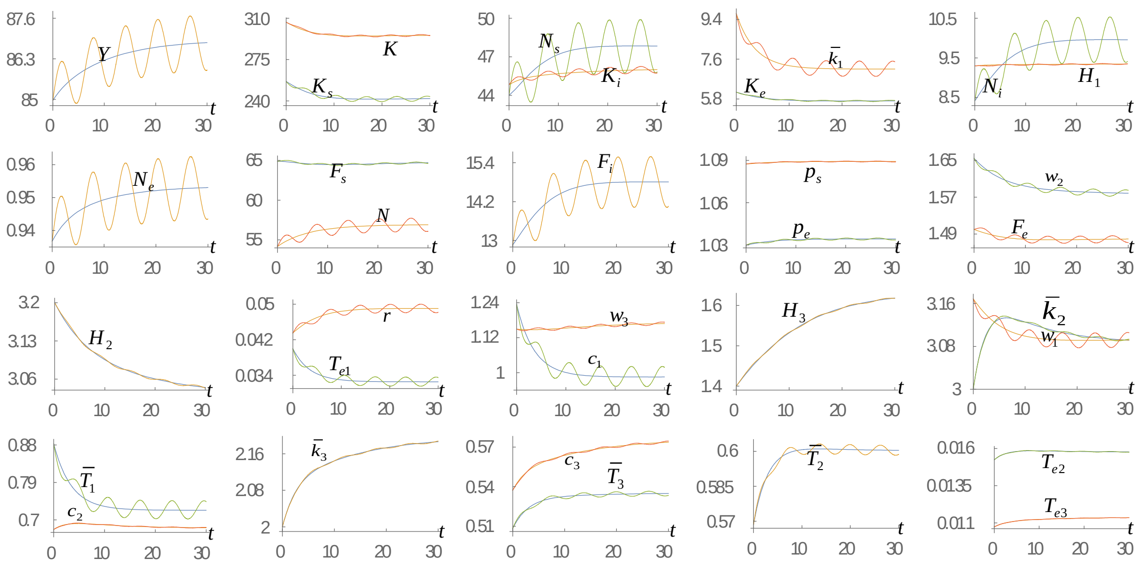

Figure 4. Fluctuations in the RC's Population

Fluctuations in the education sector's total productivity factor

We now allow the following perturbations in the total productivity of the education sector

$$A_e(t) = 0.8 + 0.05\sin(t).$$

The simulation results are plotted in Figure 5. The fluctuations in the productivity have little impact on the national output. The capital input, labor input, output level of the education sector fluctuate. The price of education also experience oscillations. The study hours of all the groups oscillate.

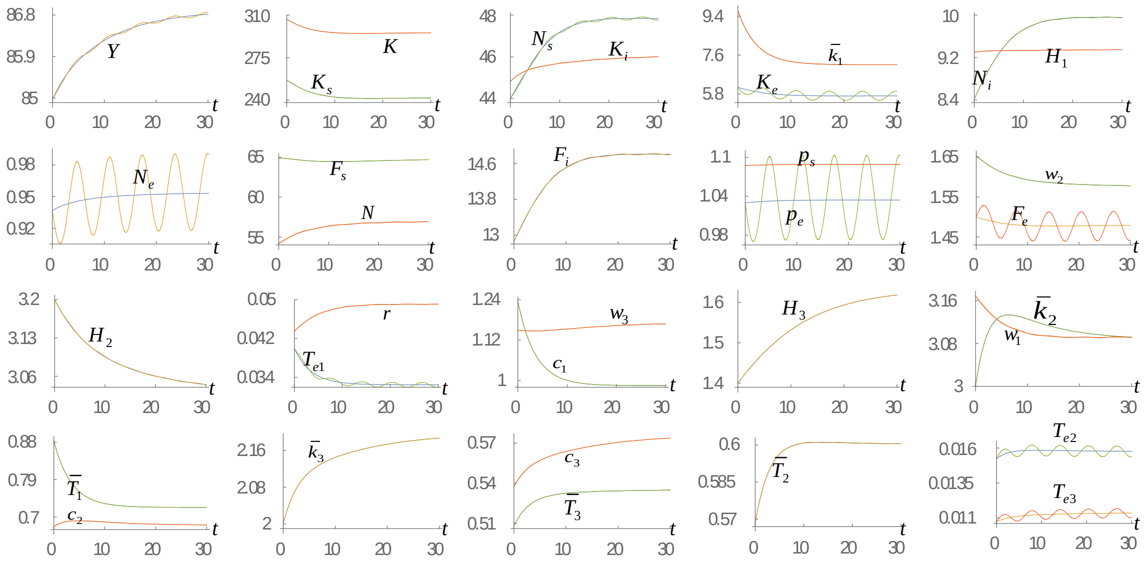

Figure 5. Fluctuations in the Education Sector's Total Productivity Factor

V. CONCLUDING REMARKS

This paper generalized the dynamic growth model with wealth accumulation and human capital accumulation proposed by Zhang (2013) by making all the parameters as time-dependent parameters. The original model is an extension of the Uzawa-Lucas model to a heterogeneous household economy with multiple ways of human capital accumulation. It synthesizes the basic ideas of the Walrasian general equilibrium theory, Arrow's learning by producing, Zhang's learning by consuming (creative learning), the neoclassical growth theory, and the Uzawa-Lucas two-sector model. The behavior of the household is described with an alternative approach to household behavior. The model describes a dynamic interdependence among wealth accumulation, human capital accumulation, and division of labor, and time distribution among leisure, education and work under perfect competition. Households are different in propensities to save, to obtain education and to consume, and in learning abilities. We simulated the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks. Our comparative dynamic analysis provides some insights into interactions between growth, inequality and education under different time-dependent shocks. We may introduce some kind of government intervention in education into the model. In this study, we don't consider public provision or subsidy of education. In the literature of education and economic growth, many models with heterogeneous households are proposed to address issues related to inequality, taxation, education policy, distribution of income and wealth, and economic growth (e.g., Bénabou, 2002; Dur and Glazer, 2008; Glomm and Kaganovich, 2008; and Nakajima and Nakamura, 2009).

Appendix: Proving Lemma

We now prove the lemma. For convenience we omit time in expressions where there is no confusion. From (6), (8), and (10) we obtain

$$z \equiv \frac{r + \delta_k}{w} = \frac{N_q}{\bar{\beta}_q K_q}, \ q = i, s, e, \qquad \text{(A1)}$$

where $\bar{\beta}_q \equiv \beta_q / \alpha_q$. From (A1), (3), and (4)

$$\frac{N_i}{\bar{\beta}_i} + \frac{N_s}{\bar{\beta}_s} + \frac{N_e}{\bar{\beta}_e} = z \sum_{j=1}^{J} \bar{k}_j \bar{N}_j . \qquad \text{(A2)}$$

Substitute (A1) into (6)

$$r(z, t) = \alpha_r z^{\beta_i} - \delta_k , \quad w(z, t) = \alpha z^{-\alpha_i} , \qquad \text{(A3)}$$

where

$$\alpha_r = \alpha_i A_i \bar{\beta}_i^{\beta_i} , \quad \alpha = \beta_i A_i \bar{\beta}_i^{-\alpha_i} .$$

We have

$$w_j\left(z, H_j, t\right) = H_j^{m_j}\ w. \qquad \text{(A4)}$$

Accordingly we have the rate of interest and the wage rates as functions of z, $\left(H_j\right)$, and t. From (7), (8), (9), and (10), we obtain

$$p_m\left(z, t\right) = \frac{\bar{\beta}_m^{\alpha_m}\ z^{\alpha_m}\ w}{\beta_m\ A_m},\ \ m = s, e. \qquad \text{(A5)}$$

From (A3) and the definitions of $\bar{y}_j$, we get

$$\bar{y}_j = \left(1 + r\right)\bar{k}_j + T_0\ w_j. \qquad \text{(A6)}$$

Insert $p_s\, c_j = \xi_j\, \bar{y}_j$ in (20)

$$\sum_{j=1}^{J} \xi_j\, \bar{N}_j\, \bar{y}_j = p_s\, F_s. \qquad \text{(A7)}$$

Substituting (A6) in (A7) yields

$$N_s = \sum_{j=1}^{J} \tilde{g}_j\, \bar{k}_j + \tilde{g}, \qquad \text{(A8)}$$

where we use $p_s\, F_s = w N_s / \beta_s$ and

$$\tilde{g}_j\left(z, t\right) \equiv \bar{r}\, \beta_s\, \xi_j\, \bar{N}_j,\ \ \bar{r}\left(z\right) \equiv \frac{1 + r}{w},\ \ \tilde{g}\left(z, \left(H_j\right), t\right) \equiv \beta_s\, T_0 \sum_{j=1}^{J} H_j^{m_j}\ \xi_j\, \bar{N}_j.$$

Insert $w_j\, \bar{T}_j = \sigma_j\, \bar{y}_j$ and $p_j\, T_{je} = \eta_j\, \bar{y}_j$ in (13)

$$T_j = T_0 - \tilde{p}_j\, \bar{y}_j, \qquad \text{(A9)}$$

where

$$\tilde{p}_j \equiv \frac{\sigma_j}{w_j} + \frac{\eta_j}{p_j}.$$

Insert (A6) in (A9)

$$T_j = \left(1 - \tilde{p}_j\, w_j\right)T_0 - \left(1 + r\right)\tilde{p}_j\, \bar{k}_j, \qquad \text{(A10)}$$

Insert (A10) in (1)

$$N = n_0 - \sum_{j=1}^{J} n_j\, \bar{k}_j, \qquad \text{(A11)}$$

where

$$n_0\left(z, \left(H_j\right), t\right) \equiv T_0 \sum_{j=1}^{J}\left(1 - \tilde{p}_j\, w_j\right) H_j^{m_j}\ \bar{N}_j,\ \ n_j\left(z, \left(H_j\right), t\right) \equiv \left(1 + r\right)\tilde{p}_j\, H_j^{m_j}\ \bar{N}_j.$$

Substituting (A8) and (A11) into (2) and (A2) yields

$$N_i + N_e = f_n,$$

$$\frac{N_i}{\bar{\beta}_i} + \frac{N_e}{\bar{\beta}_e} = f_k, \qquad \text{(A12)}$$

where

$$f_n\left(z, \left(H_j\right), \left(\bar{k}_j\right), t\right) \equiv \tilde{f}_n - \left(n_1 + \tilde{g}_1\right)\bar{k}_1,\ \ \tilde{f}_n\left(z, \left(H_j\right), \left\{\bar{k}_j\right\}, t\right) \equiv n_0 - \tilde{g} - \sum_{j=2}^{J}\left(n_j + \tilde{g}_j\right)\bar{k}_j,$$

$$f_k\left(z, \left(H_j\right), \left(\bar{k}_j\right), t\right) \equiv \left(z\, \bar{N}_1 - \frac{\tilde{g}_1}{\bar{\beta}_s}\right)\bar{k}_1 + \tilde{f}_k,$$

$$\tilde{f}_k\left(z, \left(H_j\right), \left\{\bar{k}_j\right\}, t\right) \equiv \sum_{j=2}^{J}\left(z\, \bar{N}_j - \frac{\tilde{g}_j}{\bar{\beta}_s}\right)\bar{k}_j - \frac{\tilde{g}}{\bar{\beta}_s}, \qquad \text{(A13)}$$

where $\{\bar{k}_j\} \equiv (\bar{k}_2, ..., \bar{k}_J)$. Solve (12) with N_i and N_e with the variables

$$N_i(z, (H_j), (\bar{k}_j), t) = \left(\frac{f_n}{\bar{\beta}_e} - f_k\right)\bar{\beta}, \quad N_e(z, (H_j), (\bar{k}_j), t) = \left(f_k - \frac{f_n}{\bar{\beta}_i}\right)\bar{\beta}, \tag{A14}$$

where

$$\bar{\beta} \equiv \frac{1}{1/\bar{\beta}_e - 1/\bar{\beta}_i}.$$

From (9) and (19), we have

$$\sum_{j=1}^{J} \frac{\eta_j \bar{N}_j \bar{y}_j}{p_j} = f_e N_e, \tag{A16}$$

where $T_{je} = \eta_j \bar{y}_j / p_j$ and

$$f_e(z, t) \equiv A_e \left(\frac{1}{\bar{\beta}_e z}\right)^{\alpha_e}.$$

Insert (A6) and (A14) in (A16)

$$(1 + r)\sum_{j=1}^{J} \frac{\eta_j \bar{N}_j \bar{k}_j}{p_j} = \left(f_k - \frac{f_n}{\bar{\beta}_i}\right) f_e \bar{\beta} - T_0 \sum_{j=1}^{J} \frac{\eta_j w_j \bar{N}_j}{p_j}. \tag{A17}$$

Substituting (A13) into (A17) yields

$$\bar{k}_1 = \varphi(z, \{\bar{k}_j\}, (H_j), t) \equiv \left[\left(\tilde{f}_k - \frac{\tilde{f}_n}{\bar{\beta}_i}\right) f_e \bar{\beta} - T_0 \sum_{j=1}^{J} \frac{\eta_j w_j \bar{N}_j}{p_j} - (1 + r)\sum_{j=2}^{J} \frac{\eta_j \bar{N}_j \bar{k}_j}{p_j}\right]\varphi_0, \tag{A18}$$

where

$$\varphi_0 \equiv \left[\frac{(1 + r)\eta_1 \bar{N}_1}{p_1} - \left(z \bar{N}_1 - \frac{\tilde{g}_1}{\bar{\beta}_s} + \frac{n_1 + \tilde{g}_1}{\bar{\beta}_i}\right) f_e \bar{\beta}\right]^{-1}.$$

We determine all the variables as functions of z, $\{\bar{k}_j\}$, (H_j), and t by the following procedure: r and w by (A3) $\to$ w_j by (A4) $\to$ p_e and p_s by (A5) $\to$ $\bar{k}_1$ by (A18) $\to$ N_i and N_e by (A14) $\to$ N by (A11) $\to$ N_s by (A8) $\to$ $\bar{y}_j$ by (A6) $\to$ K_m (A1) $\to$ F_i, F_s, and F_e by the definitions $\to$ $\bar{T}_j$, c_j, T_{je}, and s_j by (16) $\to K$ by (3). From this procedure, (A18), (17), and (18), we have

$$\dot{\bar{k}}_1 = \bar{\Omega}_1(z, \{\bar{k}_j\}, (H_j), t) \equiv \lambda_1 \bar{y}_1 - \varphi - \frac{\dot{\bar{N}}_1 \bar{k}_1}{\bar{N}_1}, \tag{A19}$$

$$\dot{\bar{k}}_j = \Lambda_j(z, \{\bar{k}_j\}, (H_j), t) \equiv \lambda_j \bar{y}_j - \bar{k}_j - \frac{\dot{\bar{N}}_j \bar{k}_j}{\bar{N}_j}, \quad j = 2, ..., J,$$

$$\dot{H}_j = \Omega_j\left(z, \{\bar{k}_j\}, (H_j), t\right), \quad j = 1, ..., J. \qquad \text{(A20)}$$

Taking derivatives of equation (A18) with respect to t and combining with (A20), we get

$$\dot{\bar{k}}_1 = \frac{\partial \varphi}{\partial z}\dot{z} + \sum_{j=2}^{J}\Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j} + \sum_{j=1}^{J}\Omega_j \frac{\partial \varphi}{\partial \bar{k}_j} + \frac{\partial \varphi}{\partial t}. \qquad \text{(A21)}$$

Equaling the right-hand sizes of equations (A19) and (A21), we get

$$\dot{z} = \left[\overline{\Omega}_1 - \sum_{j=2}^{J}\Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j} - \sum_{j=1}^{J}\Omega_j \frac{\partial \varphi}{\partial H_j} - \frac{\partial \varphi}{\partial t}\right]\left(\frac{\partial \varphi}{\partial z}\right)^{-1}. \qquad \text{(A22)}$$

In summary, we proved the lemma.

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